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New sequence converging towards the Euler–Mascheroni constant

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ABSTRACT

Let $\gamma = 0.577215664\ldots$ denote the Euler–Mascheroni constant, and let the sequence

$$w_n(a,b,c,d) = \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + a + \frac{b}{n} + \frac{c}{n^2} + \frac{d}{n^3}\right).$$

The main aim of this paper is to determine the values a, b, c, d which provide the fastest sequence $(w_n)_{n\geq 1}$ approximating the Euler–Mascheroni constant.

1. Introduction and preliminaries

The Euler–Mascheroni constant $\gamma = 0.577215664\ldots$ is defined as the limit of the sequence

$$D_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}).$$

Several bounds for $D_n - \gamma$ have been given in the literature. We recall some of them:

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2(n-1)} \quad \text{for } n \geq 2 \text{ ([1])},$$
$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n} \quad \text{for } n \geq 1 \text{ ([2,3])},$$
$$\frac{1-\gamma}{n} \leq D_n - \gamma < \frac{1}{2n} \quad \text{for } n \geq 1 \text{ ([4])},$$
$$\frac{1}{2n+\frac{2}{5}} < D_n - \gamma < \frac{1}{2n+\frac{1}{3}} \quad \text{for } n \geq 1 \text{ ([5])},$$
$$\frac{1}{2n+\frac{2\gamma-1}{1-\gamma}} \leq D_n - \gamma < \frac{1}{2n+\frac{1}{3}} \quad \text{for } n \geq 1 \text{ ([6,7])}.$$

The convergence of the sequence D_n to γ is very slow. Some faster approximations of the Euler–Mascheroni constants were established in [8,9,7,10–14]. For example, DeTemple [12] studied in 1993 the sequence

$$R_n = \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2}\right),$$

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and proved

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}. \tag{1}$$

Recently, Chen [7] obtained the following sharp form of the inequality (1): for all integers $n \geq 1$,

$$\frac{1}{24(n+a)^2} \leq R_n - \gamma < \frac{1}{24(n+b)^2} \tag{2}$$

with the best possible constants

$$a = \frac{1}{\sqrt{24[-\gamma + 1 - \ln(3/2)]}} - 1 = 0.55106\ldots \quad \text{and} \quad b = \frac{1}{2}.$$

The second inequality with $b = \frac{1}{2}$ in (2) results from what DeTemple [12] wrote on page 470 of the article. Chen [10] proved the second inequality in (2), and showed that $b = \frac{1}{2}$ is the best possible.

In 1997, Negoi [13] proved that the sequence

$$T_n = \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) \tag{3}$$

is strictly increasing and convergent to γ. Moreover, the author proved that

$$\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3}. \tag{4}$$

Now we define the sequence

$$w_n = \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + a + \frac{b}{n} + \frac{c}{n^2} + \frac{d}{n^3}\right). \tag{5}$$

The main aim of this paper is to find the values a, b, c, d which provide the fastest sequence $(w_n)_{n\geq 1}$ approximating the Euler–Mascheroni constant (Section 2). Also, we give the lower and upper bounds of $w_n - \gamma$ (Section 3).

Before we state and prove the main theorems, let us give some preliminary results.

The Euler–Mascheroni constant γ is deeply related to the gamma function $\Gamma(z)$ thanks to the Weierstrass formula:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^{-1} e^{z/k} \right\} \quad (z \in \mathbb{C} \setminus Z_0^-;\ Z_0^- := \{-1, -2, -3, \ldots\}).$$

The logarithmic derivative of the gamma function:

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \ln \Gamma(z) = \int_1^z \psi(t)\,dt$$

is known as the psi (or digamma) function. The successive derivatives of the psi function $\psi(z)$:

$$\psi^{(n)}(z) := \frac{d^n}{dz^n}\{\psi(z)\} \quad (n \in \mathbb{N})$$

are called the polygamma functions. The following recurrence and asymptotic formulas are well known for the psi function:

$$\psi(z+1) = \psi(z) + \frac{1}{z} \tag{6}$$

(see [15, p. 258]), and

$$\psi(z) \sim \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots \quad (z \to \infty \text{ in } |argz| < \pi) \tag{7}$$

(see [15, p. 259]). From (6) and (7), we get

$$\psi(n+1) \sim \ln n + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \cdots \quad (n \to \infty). \tag{8}$$

It is also known [15, p. 258] that

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}. \tag{9}$$

The following lemmas are needed in our present investigation.

Lemma 1 *([16,17]). If $(\lambda_n)_{n\geq 1}$ is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(\lambda_n - \lambda_{n+1}) = l \in \mathbb{R},$$

with $k > 1$, then there exists the limit:

$$\lim_{n\to\infty} n^{k-1}\lambda_n = \frac{l}{k-1}.$$

Lemma 1 gives a method for measuring the speed of convergence.

Lemma 2 *([18, Theorem 9]). Let $k \geq 1$ and $n \geq 0$ be integers. Then for all real numbers $x > 0$:*

$$S_k(2n; x) < (-1)^{k+1}\psi^{(k)}(x) < S_k(2n+1; x), \tag{10}$$

where

$$S_k(p; x) = \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{p}\left[B_{2i}\prod_{j=1}^{k-1}(2i+j)\right]\frac{1}{x^{2i+k}},$$

B_i $(i = 0, 1, 2, \ldots)$ are Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} B_i \frac{t^i}{i!}.$$

From (10), we get

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} \quad (x > 0),$$

and

$$\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} < \psi'(x+1) < \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} \quad (x > 0). \tag{11}$$

In the sequel we need the following.

Lemma 3. *Let*

$$u(x) = \frac{3840}{17}\left(\psi(x+1) - \ln\left(x + \frac{1}{2} + \frac{1}{24x} - \frac{1}{48x^2} + \frac{23}{5760x^3}\right)\right). \tag{12}$$

Then, for $x \geq 4$,

$$6\big(-u'(x)\big)^2 > 5u(x)u''(x). \tag{13}$$

Proof. We first show that for $x \geq 2$,

$$u(x) < \frac{1}{x^5} - \frac{16525}{12852x^6} + \frac{983}{8568x^7} + \frac{270239}{293760x^8}, \tag{14}$$

$$u''(x) < \frac{30}{x^7} - \frac{16525}{306x^8} + \frac{983}{153x^9} + \frac{270239}{4080x^{10}}, \tag{15}$$

$$-u'(x) > \frac{5}{x^6} - \frac{16525}{2142x^7} + \frac{983}{1224x^8} + \frac{270239}{36720x^9} - \frac{559}{32640x^{10}}. \tag{16}$$

Define the function R by

$$R(x) = u(x) - \left(\frac{1}{x^5} - \frac{16525}{12852x^6} + \frac{983}{8568x^7} + \frac{270239}{293760x^8}\right).$$

Differentiation and applying the left-hand inequality of (11) yields

$$\begin{aligned}
R'(x) &= \frac{3840}{17}\psi'(x+1) - \frac{11520}{17}\frac{1920x^4-80x^2+80x-23}{x(5760x^4+2880x^3+240x^2-120x+23)} \\
&\quad + \frac{1285200x^3-1983000x^2+206430x+1891673}{257040x^9} \\
&> \frac{3840}{17}\left(\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}+\frac{1}{42x^7}-\frac{1}{30x^9}\right) - \frac{11520}{17}\frac{1920x^4-80x^2+80x-23}{x(5760x^4+2880x^3+240x^2-120x+23)} \\
&\quad + \frac{1285200x^3-1983000x^2+206430x+1891673}{257040x^9} \\
&= \frac{3622320x^3-7010040x^2+1427190x-143543}{36720x^9(5760x^4+2880x^3+240x^2-120x+23)} \\
&= \frac{3649237+16854870(x-2)+14723880(x-2)^2+3622320(x-2)^3}{36720x^9(5760x^4+2880x^3+240x^2-120x+23)} \\
&> 0 \quad (x \geq 2).
\end{aligned}$$

This yields

$$R(x) < \lim_{x\to\infty} R(x) = \lim_{x\to\infty}\left(-\frac{559}{293760x^9}+O(x^{-10})\right) = 0 \quad (x \geq 2).$$

This proves (14).

The proofs of (15) and (16) are similar, we leave it to the reader. Consequently,

$$\begin{aligned}
6\big(-u'(x)\big)^2 - 5u(x)u''(x) &> 6\left(\frac{5}{x^6}-\frac{16525}{2142x^7}+\frac{983}{1224x^8}+\frac{270239}{36720x^9}-\frac{559}{32640x^{10}}\right)^2 \\
&\quad -5\left(\frac{1}{x^5}-\frac{16525}{12852x^6}+\frac{983}{8568x^7}+\frac{270239}{293760x^8}\right)\left(\frac{30}{x^7}-\frac{16525}{306x^8}+\frac{983}{153x^9}+\frac{270239}{4080x^{10}}\right) \\
&= \frac{1}{704741990400x^{20}}\Big(2960621408624401+10107868982530892(x-4) \\
&\quad +9894452748824996(x-4)^2+4485241143890720(x-4)^3 \\
&\quad +1060233220623200(x-4)^4+127467689187200(x-4)^5 \\
&\quad +6182190976000(x-4)^6\Big) \\
&> 0 \quad (x \geq 4).
\end{aligned}$$

Therefore, the inequality (13) holds for $x \geq 4$. □

2. An accelerated approximation to the Euler–Mascheroni constant

Theorem 1 determines the values a, b, c, d which provide the fastest sequence $(w_n)_{n\geq 1}$ approximating the Euler–Mascheroni constant.

Theorem 1. *Let $(w_n)_{n\geq 1}$ be defined by (5). For*

$$a = \frac{1}{2}, \qquad b = \frac{1}{24}, \qquad c = -\frac{1}{48}, \qquad d = \frac{23}{5760},$$

we have

$$\lim_{n\to\infty} n^6(w_n - w_{n+1}) = \frac{17}{768} \quad \textit{and} \quad \lim_{n\to\infty} n^5(w_n - \gamma) = \frac{17}{3840}.$$

The speed of convergence of the sequence $(w_n)_{n\geq 1}$ is n^{-5}.

Proof. We write the difference $w_n - w_{n+1}$ as power series in n^{-1},

$$\begin{aligned}
w_n - w_{n+1} &= \left(\frac{1}{2}-a\right)\frac{1}{n^2} + \left(-\frac{2}{3}-2b+a+a^2\right)\frac{1}{n^3} + \left(-a+3b-3c+\frac{3}{4}-\frac{3}{2}a^2-a^3+3ab\right)\frac{1}{n^4} \\
&\quad + \left(a-4b+6c-4d-\frac{4}{5}+2a^2+2a^3+a^4-6ab+4ac+2b^2-4ba^2\right)\frac{1}{n^5}
\end{aligned}$$

$$+\left(-a+5b-10c+10d+\frac{5}{6}-\frac{5}{2}a^2-\frac{10}{3}a^3-\frac{5}{2}a^4+10ab-10ac\right.$$
$$\left.+\ 5ad-5b^2+10ba^2-a^5+5cb-5ca^2-5ab^2+5ba^3\right)\frac{1}{n^6}+O\left(\frac{1}{n^7}\right). \tag{17}$$

According to Lemma 1, we have four parameters a, b, c, d, which produce the fastest convergence of the sequence from (17)

$$\begin{cases} \frac{1}{2}-a=0 \\ -\frac{2}{3}-2b+a+a^2=0 \\ -a+3b-3c+\frac{3}{4}-\frac{3}{2}a^2-a^3+3ab=0 \\ a-4b+6c-4d-\frac{4}{5}+2a^2+2a^3+a^4-6ab+4ac+2b^2-4ba^2=0, \end{cases}$$

namely, if

$$a=\frac{1}{2}, \qquad b=\frac{1}{24}, \qquad c=-\frac{1}{48}, \qquad d=\frac{23}{5760}.$$

Thus, we have

$$w_n-w_{n+1}=\frac{17}{768n^6}+O\left(\frac{1}{n^7}\right).$$

By using Lemma 1, we obtain the assertion of Theorem 1. $\square$

In the following table, we present numerical computations to confirm the superiority of our sequence

$$w_n=\sum_{k=1}^{n}\frac{1}{k}-\ln\left(n+\frac{1}{2}+\frac{1}{24n}-\frac{1}{48n^2}+\frac{23}{5760n^3}\right) \tag{18}$$

over the sequence $(T_n)_{n\ge1}$ defined by (3).

n	$\lvert T_n-\gamma\rvert$	$\lvert w_n-\gamma\rvert$
5	1.44×10^{-4}	1.06×10^{-6}
10	1.94×10^{-5}	3.8×10^{-8}
15	5.89×10^{-6}	5×10^{-9}

We now propose the following.

Open Problem 1. For a given positive integer p, please find the constants a_i $(i=0,1,2,\ldots,p)$ such that

$$\sum_{k=1}^{n}\frac{1}{k}-\ln\left(n+\sum_{i=0}^{p}a_in^{-i}\right)$$

is the *fastest* sequence which would converge to γ.

3. Bounds of $w_n-\gamma$

Theorem 2 gives the higher order estimate of $w_n-\gamma$.

Theorem 2. *Let $(w_n)_{n\ge1}$ be defined by (18). For all integers $n\ge1$, then*

$$\frac{17}{3840\left(n+\frac{4808}{12852}\right)^5}<w_n-\gamma<\frac{17}{3840\left(n+\frac{3305}{12852}\right)^5}. \tag{19}$$

The constant $\frac{3305}{12852}$ in the upper bound is the best possible.

Proof. The lower bound is obtained by considering the function f defined for $x\ge0$ by

$$f(x)=\psi(x+1)-\ln\left(x+\frac{1}{2}+\frac{1}{24x}-\frac{1}{48x^2}+\frac{23}{5760x^3}\right)-\frac{17}{3840\left(x+\frac{4808}{12852}\right)^5}.$$

Differentiation and applying the right-hand inequality of (11) yields

$$\begin{aligned}
f'(x) &= \psi'(x+1) - \frac{3(1920x^4 - 80x^2 + 80x - 23)}{x(5760x^4 + 2880x^3 + 240x^2 - 120x + 23)} + \frac{623436205544413176365 1}{256(3213x + 1202)^6} \\
&< \left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}\right) - \frac{3(1920x^4 - 80x^2 + 80x - 23)}{x(5760x^4 + 2880x^3 + 240x^2 - 120x + 23)} \\
&\quad + \frac{6234362055444131763651}{256(3213x + 1202)^6} \\
&= -\frac{g(x)}{26880x^7(5760x^4 + 2880x^3 + 240x^2 - 120x + 23)(3213x + 1202)^6},
\end{aligned}$$

where

$$\begin{aligned}
g(x) &= 1297861464349251775184712794493 7 + 1251813353301033145589160212466 69(x-3) \\
&\quad + 296413571676867268399836173089597(x-3)^2 + 348465635957236559695715591514249(x-3)^3 \\
&\quad + 246671824769585407139702161152595(x-3)^4 + 113774659077906339749916168085191(x-3)^5 \\
&\quad + 35229693095638664813593213308687(x-3)^6 + 7302181671341281269281614434291(x-3)^7 \\
&\quad + 975527033001316411236158499192(x-3)^8 + 76156834112776483191337327920(x-3)^9 \\
&\quad + 2645716565458596717393163200(x-3)^{10} > 0 \quad \text{for } x \geq 3.
\end{aligned}$$

Therefore, $f'(x) < 0$ for $x \geq 3$.

Straightforward calculation produces $f(1) = 8.83 \times 10^{-8} \ldots$ and $f(2) = 0.00000596\ldots, f(3) = 0.000001040\ldots$, thus, the sequence

$$f(n) = \psi(n+1) - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}\right) - \frac{17}{3840\left(n + \frac{1}{2}\right)^5}$$

is strictly decreasing for all integers $n \geq 1$. This leads to

$$f(n) > \lim_{n\to\infty} f(n) = 0,$$

by using the asymptotic formula (8). Hence, the left-hand side of inequality (19) is valid for all $n \in \mathbb{N}$.

The upper bound is obtained by considering the function F defined for $x \geq 0$ by

$$F(x) = \psi(x+1) - \ln\left(x + \frac{1}{2} + \frac{1}{24x} - \frac{1}{48x^2} + \frac{23}{5760x^3}\right) - \frac{17}{3840\left(x + \frac{3305}{12852}\right)^5}.$$

Differentiation and applying the left-hand inequality of (11) yields

$$\begin{aligned}
F'(x) &= \psi'(x+1) - \frac{3(1920x^4 - 80x^2 + 80x - 23)}{x(5760x^4 + 2880x^3 + 240x^2 - 120x + 23)} + \frac{9949792887106108218416}{(12852x + 3305)^6} \\
&> \left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9}\right) \\
&\quad - \frac{3(1920x^4 - 80x^2 + 80x - 23)}{x(5760x^4 + 2880x^3 + 240x^2 - 120x + 23)} + \frac{99749792887106108218416}{(12852x + 3305)^6} \\
&= \frac{G(x)}{210x^9(5760x^4 + 2880x^3 + 240x^2 - 120x + 23)(12852x + 3305)^6},
\end{aligned}$$

where

$$\begin{aligned}
G(x) &= 8045748956389867461666669810484 3 + 7332748062831533703735573596354 52(x-2) \\
&\quad + 243173731928692208635828339250927 5(x-2)^2 + 437649271125672321581766868014108 0(x-2)^3 \\
&\quad + 495765099066766891668173091555803 5(x-2)^4 + 378708411241474774478554316101353 6(x-2)^5 \\
&\quad + 201283386597106123158585346826445 1(x-2)^6 + 749424061132339705160399058376305(x-2)^7 \\
&\quad + 192406966850086960927283274852360(x-2)^8 + 32541290855913855750477492934800(x-2)^9 \\
&\quad + 3270220576280741633688872084736(x-2)^{10} + 148181533411316889893879957760(x-2)^{11} \\
&> 0 \quad \text{for } x \geq 2,
\end{aligned}$$

Therefore, $F'(x) > 0$ for $x \geq 2$.

Direct calculation produces $F(1) = -0.000506058\ldots$ and $F(2) = -0.000010902\ldots$, thus, the sequence

$$F(n) = \psi(n+1) - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}\right) - \frac{17}{3840\left(n+\frac{1}{4}\right)^5}$$

is strictly increasing for all integers $n \geq 1$. This leads to

$$F(n) < \lim_{n\to\infty} F(n) = 0,$$

by using the asymptotic formula (8). Hence, the right-hand side of inequality (19) is valid for all $n \in \mathbb{N}$.

Write the right-hand side of inequality (19) as

$$\frac{1}{\sqrt[5]{\frac{3840}{17}\left(\psi(n+1) - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}\right)\right)}} - n > \frac{3305}{12852}.$$

It is easy to see that

$$\begin{aligned}
\ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}\right) &= \ln n + \ln\left(1 + \frac{1}{2n} + \frac{1}{24n^2} - \frac{1}{48n^3} + \frac{23}{5760n^4}\right) \\
&= \ln n + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{1}{2n} + \frac{1}{24n^2} - \frac{1}{48n^3} + \frac{23}{5760n^4}\right)^k \\
&= \ln n + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{17}{3840n^5} + \frac{143}{82944n^6} \\
&\quad - \frac{983}{1935360n^7} + O\left(\frac{1}{n^8}\right).
\end{aligned} \tag{20}$$

By using the asymptotic formulas (8) and (20), we conclude that

$$\frac{3840}{17}\left(\psi(n+1) - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}\right)\right) = \frac{1}{n^5} - \frac{16525}{12852n^6} + \frac{983}{8568n^7} + O\left(\frac{1}{n^8}\right). \tag{21}$$

From the asymptotic formulas (21), we get

$$\frac{1}{\sqrt[5]{\frac{3840}{17}\left(\psi(n+1) - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}\right)\right)}} - n = \frac{\frac{3305}{12852} + O(n^{-1})}{1 + O(n^{-1})} \to \frac{3305}{12852} \quad (n\to\infty). \tag{22}$$

This means that the constant $\frac{3305}{12852}$ in the upper bound is the best possible. The proof of Theorem 2 is complete. □

In view of the inequality (19) it is natural to ask: What is the smallest number α and what is the largest number β such that the inequality

$$\frac{17}{3840(n+\alpha)^5} \leq w_n - \gamma \leq \frac{17}{3840(n+\beta)^5}$$

holds for all integers $n \geq 1$? The following Theorem 3 answers this question.

Theorem 3. *For all integers $n \geq 1$,*

$$\frac{17}{3840(n+\alpha)^5} \leq w_n - \gamma < \frac{17}{3840(n+\beta)^5} \tag{23}$$

with the best possible constants

$$\alpha = \frac{1}{\sqrt[5]{\frac{3840}{17}\left(1 - \gamma - \ln\left(\frac{8783}{5760}\right)\right)}} - 1 = 0.374078\ldots \quad \text{and} \quad \beta = \frac{3305}{12852} = 0.257158\ldots.$$

Proof. The inequality (23) can be written as

$$\alpha \geq \frac{1}{\sqrt[5]{\frac{3840}{17}\left(\psi(n+1) - \ln\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3}\right)\right)}} - n > \beta.$$

In order to prove (23) we define the function P by

$$P(x)=\big(u(x)\big)^{-1/5}-x,$$

where $u(x)$ is as in Lemma 3.

Differentiation yields

$$P'(x)=-\frac{u'(x)}{5\big(u(x)\big)^{6/5}}-1\quad\text{and}\quad P''(x)=\frac{6\big(-u'(x)\big)^2-5u(x)u''(x)}{25\big(u(x)\big)^{11/5}}.$$

By (13), we obtain $P''(x)>0$ for $x\geq 4$. This implies that

$$P'(x)<\lim_{x\to\infty}P'(x)=\lim_{x\to\infty}\left(-\frac{48298367}{275289840x^2}+O(x^{-3})\right)=0\quad(x\geq 4). \tag{24}$$

From (24) and $P(1)=0.204146\ldots$, $P(2)=0.014605\ldots$, $P(3)=0.002521\ldots$, $P(4)=0.000682\ldots$, we conclude that the sequence

$$P(n)=\frac{1}{\sqrt[5]{\frac{3840}{17}\left(\psi(n+1)-\ln\left(n+\frac{1}{2}+\frac{1}{24n}-\frac{1}{48n^2}+\frac{23}{5760n^3}\right)\right)}}-n\quad(n\in\mathbb{N})$$

is strictly decreasing. This leads to

$$\lim_{n\to\infty}P(n)<P(n)\leq P(1)=\frac{1}{\sqrt[5]{\frac{3840}{17}\left(1-\gamma-\ln\left(\frac{8783}{5760}\right)\right)}}-1=0.374078\ldots.$$

It remains to prove that

$$\lim_{n\to\infty}P(n)=\frac{3305}{12852}. \tag{25}$$

The limit relation (25) has been proved in Theorem 2 (see (22)). The proof of Theorem 3 is complete. □

Remark 1. Some calculations in this work were performed by using the Maple software for symbolic calculations.

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